hep-ph/9906222
Edinburgh 99/5
RM3-TH/99-1
DFTT 27/99

The Small x Behaviour of Altarelli-Parisi Splitting Functions

Richard D. Ball*

Department of Physics and Astronomy
University of Edinburgh, EH9 3JZ, Scotland

Stefano Forte†

I.N.F.N., Sezione di Roma III
Via della Vasca Navale 84, I-00146 Rome, Italy

Abstract

We extract the small x asymptotic behaviour of the Altarelli-Parisi splitting functions from their expansion in leading logarithms of $1/x$. We show in particular that the nominally next-to-leading correction extracted from the Fadin-Lipatov kernel is enhanced asymptotically by an extra $\ln \frac{1}{x}$ over the leading order. We discuss the origin of this problem, its dependence on the choice of factorization scheme, and its all-order generalization. We derive necessary conditions which must be fulfilled in order to obtain a well behaved perturbative expansion, and show that they may be satisfied by a suitable reorganization of the original series.

June 1999

* Royal Society University Research Fellow
† On leave from INFN, Sezione di Torino, Italy

The inclusive structure function $F_2(x, Q^2)$ has been determined with extraordinary accuracy down to very small x by recent experiments at the HERA collider [1]. For $Q^2 \gtrsim 1\mathrm{GeV}^2$ the x and Q^2 dependence of the data is in complete agreement with that predicted by the next-to-leading order (NLO) Altarelli-Parisi evolution equations. At small x and large Q^2, the evolution equations are dominated by the small x singularities of the Altarelli-Parisi splitting functions [2], and retaining only the singularities in the LO and NLO splitting functions yields an excellent approximation to the full solution in the HERA region [3].

As we go to higher orders in α_s the splitting functions become more and more singular, and these higher order singularities might be expected to become dominant at small enough x. It thus appears reasonable to try to improve the description of small x evolution by supplementing the usual leading-order splitting functions with contributions which sum all leading logs of x (LLx) to all orders in α_s, *i.e.* all terms of the form $(\alpha_s \log x)^n$. Likewise, the NLO splitting functions can be supplemented by a summation of next-to-leading log x contributions (*i.e.* all terms of the form $\alpha_s(\alpha_s \log x)^n$, and so on (the 'double-leading expansion' [4]).

However, as is by now well known [5], such attempts are unsuccessful: essentially, the data in the HERA region are so accurately described by plain NLO evolution (and the small x approximation to it) that any further "improvement" would spoil this agreement unless its effects were extremely small. Furthermore, the recent determination [6,7,8] of the subleading corrections to LLx evolution has shown that NLLx contributions are extremely large, and in fact grow faster as $x \to 0$ than their LLx counterparts [9,10]. Hence, contrary to naive expectations, the double leading expansion does not appear to be stable at small x. A deeper understanding of the all-order behaviour of splitting functions at small x is required.

In this letter we will determine the small x behaviour of the Altarelli-Parisi splitting functions order by order in LLx, NLLx,. . . . All our discussion will be based on a formal perturbative treatment of the small x expansion: at NLLx we retain only the contributions to small x evolution calculated in [6], systematically discarding any terms which are formally NNLLx. This approach allows us to isolate the reason for the asymptotic breakdown of the small x expansion. This turns out to be unrelated to various problems discussed elsewhere, such as the unphysical behaviour of the solutions found in [11,12] and the running coupling resummation effects discussed in [13,14]. We find that the instability observed at NLLx is not some peculiar feature of the calculation of ref.[6], but is completely generic, probably persisting to all orders. Formally subleading contributions to the splitting function are not suppressed by powers of α_s, but rather grow faster and faster at small x. We derive conditions on the small x evolution kernel which are necessary for stable evolution, and show that they may be satisfied by a suitable reorganization of the perturbative expansion. Such a resummation introduces an a priori undetermined parameter which describes the resummed all-order small-x behaviour of the structure functions.

The small x behaviour of splitting functions $P(x)$ is most easily studied by considering their Mellin transforms, the anomalous dimensions $\gamma(N) \equiv \int_0^1 dx\, x^N P(x)$. The leading small x behaviour is then found by expanding $\gamma(N)$ about its rightmost singularity in the complex N plane, which in the singlet sector is located at $N = 0$ (and at $N = -1$ in

the nonsinglet sector, which is thus suppressed by a power of x and hence asymptotically negligible). The anomalous dimensions in the singlet sector are given by a two-by-two matrix; however only one of the two eigenvalues of the matrix is singular at $N = 0$ at leading order (and to all orders in appropriate factorization schemes). It is thus sufficient to concentrate on this leading eigenvalue and its associated eigenvector $G_N(Q^2)$, the Mellin transform of the distribution function $G(x, Q^2)$, which satisfies the evolution equation

$$\frac{d}{dt} G_N(Q^2) = \gamma(N; a) G_N(Q^2), \tag{1}$$

where $t \equiv \ln(Q^2/\Lambda^2)$ and for future convenience we write $a(t) \equiv \alpha_s(t)/2\pi$.

The general structure of the anomalous dimension in the small x expansion is

$$\gamma(N; a) = \gamma_0(a/N) + a\gamma_1(a/N) + \dots \tag{2}$$

$$\gamma_k \left(\frac{a}{N}\right) \equiv \sum_{n=1}^{\infty} A_k^{(n)} \left(8N_c \ln 2 \frac{a}{N}\right)^n . \tag{3}$$

where the coefficents $A_k^{(n)}$ have been normalized such that the radius of convergence of the series is one. The associated splitting functions P_k are immediately obtained by inverse Mellin transformation of γ_k:

$$P(x; a) = \frac{a}{x} \left[P_0(\Xi) + aP_1(\Xi) + \dots\right] \tag{4}$$

$$P_k(\Xi) = \sum_{n=1}^{\infty} A_k^{(n)} \frac{\Xi^{n-1}}{(n-1)!}; \tag{5}$$

where $\xi = \ln \frac{1}{x}$ and we define

$$\Xi \equiv (2C_A)(4 \ln 2) \tfrac{\alpha_s}{2\pi} \xi = 8N_c \ln 2\, a\xi. \tag{6}$$

Subsequent terms γ_k, P_k in the expansions (3),(5) of the anomalous dimension and splitting function sum the leading, next-to-leading,... logarithms of $\frac{1}{x}$. They can therefore be determined [15,16] from knowledge of the respective leading, next-to-leading,... QCD high-energy asymptotics, as given by leading log x evolution, which is in turn controlled by an equation of the form

$$\frac{d}{d\xi} G_M(x) = a\chi(M; a) G_M(x), \tag{7}$$

where the Mellin variable M is defined by

$$G_M(x) \equiv \int_0^{\infty} \frac{dQ^2}{Q^2} \left(\frac{Q^2}{\Lambda^2}\right)^{-M} G(x, Q^2) \tag{8}$$

and the anomalous dimension $\chi(M; a)$ admits a perturbative expansion

$$\chi(M; a) = \chi_0(M) + a\chi_1(M) + \dots \quad . \tag{9}$$

The leading-order term $\chi_0(M) = 2N_c[2\psi(1) - \psi(M) - \psi(1-M)]$ is well known [17], while the next-to-leading term χ_1 has been determined only recently [6].[1] Note that since structure functions scale in the $Q^2 \to \infty$ limit and drop linearly with Q^2 as $Q^2 \to 0$ the leading-twist physical region corresponds to $0 < M < 1$.

The k-th order small x anomalous dimensions $\gamma_k\left(\frac{a}{N}\right)$ eq. (2) can be determined from $\chi_0, \ldots, \chi_k$ eq. (9) by matching the solutions to the respective evolution equations. Assume for the moment that the coupling is fixed. Solving eq. (7) by Mellin transform with respect to N, we find

$$G_{NM} = \frac{G^0(M)}{N - a\chi(M;a)}, \tag{10}$$

where $G^0(M)$ is the boundary condition at $\xi = 0$. In order to compare to eq. (1), invert the M-Mellin at large Q^2:

$$G_N(Q^2) = \frac{G^0[M_p(N;a)]}{(-a\chi'[M_p(N;a)])} e^{M_p(N;a)t}, \tag{11}$$

where $M_p(N;a)$ is the position of the rightmost pole of G_{NM} in the M-plane in the physical region $0 < M < 1$. It follows that the solutions to the evolution equations (1) and (7) coincide only if $\gamma = M_p$, *i.e.* if their anomalous dimensions satisfy the 'duality' relation [16]

$$\chi[\gamma(N;a);a] = \frac{N}{a}. \tag{12}$$

Expansion of eq. (12) in powers of a keeping a/N fixed gives a set of relations which determine γ_k perturbatively order by order in terms of the perturbative expansion of χ:

$$\chi_0[\gamma_0(a/N)] = \frac{N}{a} \tag{13}$$

$$\gamma_1\,(a/N) = -\frac{\chi_1[\gamma_0(a/N)]}{\chi_0{}'[\gamma_0(a/N)]} \tag{14}$$

$$\gamma_2(a/N) = -\frac{\chi_2(\chi_0{}')^2 - \chi_1\chi_1{}'\chi_0{}' + \frac{1}{2}(\chi_1)^2\chi_0{}''}{(\chi_0{}')^3}, \tag{15}$$

and so forth, where the prime indicates differentiation with respect to M, and in the last equation all χ-functions have argument $\gamma_0(a/N)$. Eq. (13) should be viewed as an implicit equation for γ_0, with $\chi_0(M)$ evaluated in the physical region $0 < M < 1$.

The derivation presented so far holds at fixed coupling α_s. If the coupling runs with Q^2 in both equations, it is sufficient to include the running up to order α_s^k when computing the anomalous dimensions to k-th order in the LLx expansion. The contributions to the kernel

[1] Note that here we have adopted a different set of normalization conventions to those used in ref. [6], where the right hand side of eq. (9) is written as $2N_c(\chi + \frac{N_c\alpha_s}{4\pi}\tilde{\delta})$: our χ_0 is the same as their $2N_c\chi$, while our χ_1 is the same as their $N_c^2\tilde{\delta}$.

on the right hand side of eq.(7) are then found by replacing α_s by differential operators: for example, to NLLx it is sufficient to let

$$a \to a\left(1 + b_0 a \frac{d}{dM}\right), \tag{16}$$

where $b_0 = \frac{1}{2}(\frac{11}{3}N_c - \frac{2}{3}n_f)$. Solving (7) as before, taking care to treat all NLLx terms (including the derivative term) as perturbations, now gives

$$G_{NM} = \left[\frac{1}{1-\frac{a}{N}\chi_0} + a\frac{a}{N}\Big(\frac{\chi_1 + b_0\chi_0(\ln\chi_0 G_0)'}{(1-\frac{a}{N}\chi_0)^2} + \frac{b_0\chi_0\frac{a}{N}\chi_0'}{(1-\frac{a}{N}\chi_0)^3}\Big) + \dots\right]\frac{G^0(M)}{N}. \tag{17}$$

Inverting the M-Mellin, and again comparing with the solution of eq. (1) with the running coupling expanded at NLLx, we see that the solutions match if in place of (14) we now have

$$\gamma_1(a/N) = -\frac{1}{\chi_0{}'(\gamma_0)}\left(\chi_1(\gamma_0) + \tfrac{1}{2}b_0\frac{\chi_0(\gamma_0)\chi_0''(\gamma_0)}{\chi_0'(\gamma_0)}\right) \tag{18}$$

$$= -\frac{\chi_1(\gamma_0)}{\chi_0{}'(\gamma_0)} - \frac{1}{a}\frac{d}{dt}\ln\sqrt{-\chi_0'(\gamma_0)}, \tag{19}$$

since to leading order $da/dt = -b_0 a^2$, while $a\chi_0(\gamma_0) = N$ so $\partial\gamma_0/\partial\ln a = -\chi_0/\chi_0'$. The expression (19) was obtained in [18] by solving eq.(7) with running coupling exactly, and then inverting the Mellin by a saddle point argument consistently at NLLx.

When the coupling runs there is however a further ambiguity related to the choice of factorization scheme. Under a LLx change in the normalization of $G_M(\xi)$, *i.e.* $G_M(\xi) \to u(M)G_M(\xi)$, the NLLx kernel changes according to

$$\chi_1(M) \to \chi_1(M) - b_0\chi_0(M)\frac{d}{dM}\ln u(M). \tag{20}$$

From eq.(19) the anomalous dimension changes as

$$\gamma \to \gamma + \frac{d}{dt}\ln u(\gamma_0). \tag{21}$$

This is the same as the NLLx shift in the anomalous dimension induced by the LLx scheme change $G_N(t) \to u(a(t)/N)G_N(t)$, where $u(a/N) = 1 + \sum_1^\infty u_n(a/N)^n$ [19]. Note that a NLLx scheme change only affects the anomalous dimension at NNLLx: the mismatch is due to the coupling running with Q^2 while the logarithms are ordered in x. It follows that knowledge of the leading order coefficient function [20] is sufficient for a consistent calculation at NLLx.

In order to determine the NLLx anomalous dimension γ_1 in $\overline{\mathrm{MS}}$ factorization[2] from the Fadin-Lipatov kernel χ_1^{FL} (eqns.(14)& (22) of ref.[6]) two further adjustments are required.

[2] Note that the large eigenvalue of evolution at small x is invariant under the usual (NLLQ) scheme changes: it follows that for our purposes the $\overline{\mathrm{MS}}$ scheme is the same as the DIS scheme, and indeed we will use these two notations interchangeably.

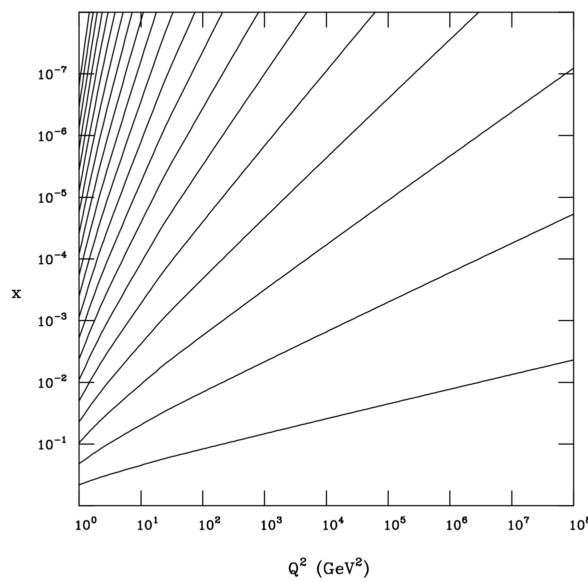

Figure 1: Contours of constant $\Xi = 1, 2, \ldots, 20$ (from bottom to top) in the $x - Q^2$ plane.

First, the kernel χ_1 for the small x evolution of the distribution $G_M(\xi) \equiv M^{-1} g_M(\xi)$ is related to that for the unintegrated distribution $g_M(\xi)$ employed in [6] by $\chi_1(M) = \chi_1^{\rm FL}(M) + b_0 \chi_0(M)/M$. This may be thought of as a LLx scheme change with $u(M) = M$. Second, the correct expression for γ_1 with $\overline{\rm MS}$ factorization requires a further scheme change with $u(M) = R(M)$, where $R(\gamma_0(a/N))) \equiv R_N(\alpha_s)$ is calculated in ref.[20].[3] The $\overline{\rm MS}$ NLLx anomalous dimension is thus

$$\gamma_1(a/N) = -\frac{\chi_1^{FL}(\gamma_0)}{\chi_0{}'(\gamma_0)} + \frac{1}{a}\frac{d}{dt} \ln R(\gamma_0)\gamma_0 \sqrt{-\chi_0'(\gamma_0)} \tag{22}$$

$$= -\frac{1}{\chi_0{}'(\gamma_0)} \left(\chi_1^{FL}(\gamma_0) + b_0 N_c((2\psi'(1) - \psi'(\gamma_0) - \psi'(1-\gamma_0)) + \tfrac{1}{4N_c^2}\chi_0(\gamma_0)^2) \right), \tag{23}$$

where in the second line we used eq.(B.18) of [20] for $\partial \ln R/\partial\gamma_0$.

It is clear from eqns.(22) & (23) that at NLLx the effect of the running of α_s, eq.(19), can be entirely absorbed into a shift of the NLLx anomalous dimension γ_1 equivalent to a choice of factorization scheme. It turns out that this property persits to higher orders in the small x expansion. We may thus view running coupling effects as a contribution to χ: henceforth we assume that χ incorporates the running coupling effects at the appropriate order in the small x expansion, so that the anomalous dimension γ is given by the duality relation (12).

[3] To see this compare the result for γ_{qg} in $\overline{\rm MS}$ obtained in [20] with that derived in [21]: their ratio must be u. Alternatively, the explicit derivation given in Appendix B of [20] of the anomalous dimension from the dimensionally regularized kernel may be extended to NLLx: this gives the same result [22].

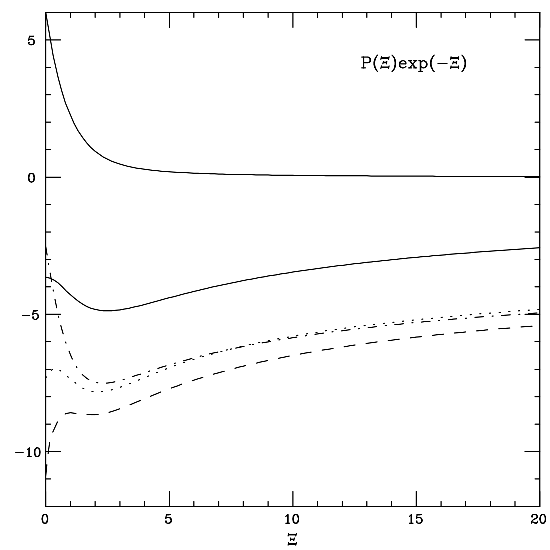

Figure 2: The LLx splitting function $P_0(\Xi)$ (solid, positive) and the NLLx splitting function $\frac{1}{2\pi}P_1(\Xi)$ (negative), computed numerically (using eqns. (5),(13),(19),(23) and χ_1^{FL} from ref.[6]), and renormalised by a factor $e^{-\Xi}$, in various factorization schemes: DIS (or $\overline{\mathrm{MS}}$, see footnote 2) (solid), Q_0-DIS [21] (dotted), SDIS [23] (dashed) and GDIS [24,19] (dot-dashed).

As is well known, the BFKL kernel $\chi_0(M)$ is symmetric about $M = \frac{1}{2}$, where it has a minimum, with $\chi_0(\frac{1}{2}) = 8N_c \ln 2$. It follows that γ_0, viewed as a series of powers of $8N_c \ln 2 \frac{a}{N}$ (eq.(3)), has unit radius of convergence. All higher order γ_k will have the same radius of convergence in any factorization scheme in which all $\chi_k(M)$ are free of singularities for $0 < M < \frac{1}{2}$. The fact that the series for the anomalous dimension has finite radius of convergence is important because it shows that the corresponding splitting function has infinite radius of convergence as a power series in Ξ, and can thus be used down to arbitrarily small x, provided a large enough number of terms is included [4]. In the asymptotic region where $A_k^{(n+1)}/A_k^{(n)} \sim 1$ the number of terms which must be included is of order $k_c(x, Q^2)$, where k_c is the solution of the implicit equation $\Xi^{k_c}/k_c! \sim 1$, whence $k_c \sim 2.7\Xi$ for $\Xi \gtrsim 1$. It is apparent that already in the HERA region, where Ξ can be as large as ten (see fig. 1) a large (though finite) number of terms should be included.[4]

Since each contribution P_k to the small x expansion (4) of the splitting function sums up leading logs of x to all orders, each subsequent order in the expansion appears to be of order α_s compared to the previous order. It was thus natural to conjecture [15,20,4] that evolution at small x can be accurately described by truncating the expansion to finite order, provided only that α_s is small enough (*i.e.* Q^2 is large enough). As discussed in the

[4] It is interesting to contrast this with the familiar case of large x (Sudakov) resummation, where the series of contributions to be resummed in the anomalous dimension is divergent and can only be treated by Borel resummation: for meaningful results an infinite number of terms must be included whenever $\alpha_s \ln(1-x) \sim 1$.

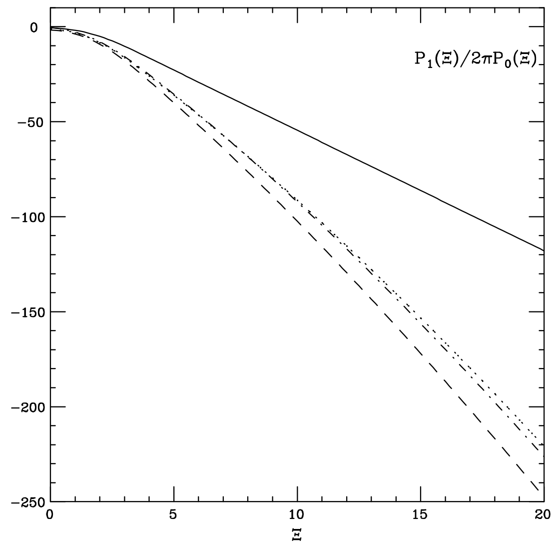

Figure 3: The ratio $\frac{1}{2\pi}P_1(\Xi)/P_0(\Xi)$ of NLLx and LLx splitting functions, computed as in fig. 2, in the same factorization schemes .

introduction, however, this conjecture does not seem in agreement with phenomenology [5], nor with the numerical comparison of the NLLx splitting function P_1 to the LLx P_0 [9,10]: as may be seen from Fig. 2, P_1 is large and negative, and furthermore the size of the ratio of P_1 to P_0 (see fig. 3) increases rapidly at small x, *i.e.* as Ξ increases. It is apparent from the plot that the ratio $\frac{1}{2\pi}P_1/P_0$ is much greater than $1/\alpha_s(Q^2) \lesssim 10$ throughout most of the range of Ξ relevant for HERA and the LHC. This in itself explains the failure of the phenomenology based on a leading-order truncation of the expansion at HERA: the small x expansion breaks down for any reasonable value of α_s [9].

To understand this result we will now consider analytically the asymptotic behaviour of P_k at large values of Ξ. The asymptotic behaviour of the splitting functions may be determined through their definition as the inverse Mellin transform of the anomalous dimension γ [13]: at LLx

$$P_0(\Xi) = \int_{-i\infty}^{i\infty} \frac{dN}{2\pi i a}\, e^{\xi N}\gamma_0(a/N) = -\int_{-i\infty}^{i\infty} \frac{d\gamma_0}{2\pi i}\, e^{\xi a \chi_0(\gamma_0)}\gamma_0\chi_0'(\gamma_0), \tag{24}$$

where we have used eq. (13) to change integration variable $dN = a\chi_0'(\gamma_0)d\gamma_0$. The asymptotic expansion of $P_0(\Xi)$ as $\xi \to \infty$ can then be determined by the saddle point method: a straightforward computation, remembering that the only real minimum of $\chi_0(\gamma_0)$ in the range $0 < \gamma_0 < 1$ is at $\gamma_0 = \frac{1}{2}$ leads to

$$P_0(\Xi) \underset{\Xi\to\infty}{\sim} \left(\frac{\chi_0(\frac{1}{2})}{2\pi\chi_0''(\frac{1}{2})}\right)^{1/2} \frac{\chi_0(\frac{1}{2})}{\Xi^{3/2}}\, e^{\Xi}\left[1 + \frac{\chi_0(\frac{1}{2})\chi_0''''(\frac{1}{2})}{8(\chi_0''(\frac{1}{2}))^2}\frac{1}{\Xi} + O\left(\frac{1}{\Xi^2}\right)\right], \tag{25}$$

where Ξ is given by eq. (6), while the n-th derivatives $\chi_0^{(n)}(\frac{1}{2}) = 4N_c\, n!(2^{n+1}-1)\zeta(n+1)$ when n is even (all odd derivatives vanish).

The asymptotic behaviour of the next-to-leading correction P_1 can be determined in a similar way:

$$\begin{aligned} P_1(\Xi) &= -\int_{-i\infty}^{i\infty} \frac{dN}{2\pi i a}\, e^{\xi N} \frac{\chi_1[\gamma_0(a/N)]}{\chi_0'[\gamma_0(a/N)]} = -\int_{-i\infty}^{i\infty} \frac{d\gamma_0}{2\pi i} e^{\xi a \chi_0(\gamma_0)} \chi_1(\gamma_0) \\ &\underset{\Xi\to\infty}{\sim} \left(\frac{\chi_0(\frac{1}{2})}{2\pi\chi_0''(\frac{1}{2})}\right)^{1/2} \frac{\chi_1(\frac{1}{2})}{\Xi^{1/2}} e^{\Xi} \left[1 + O\left(\frac{1}{\Xi}\right)\right], \end{aligned} \tag{26}$$

where in the last step we have assumed $\chi_1(M)$ to be regular at $M = \frac{1}{2}$. This requirement can always be achieved by choice of factorization scheme: in particular $\chi_1^{FL}(\frac{1}{2}) = -71.64N_c^2 - 0.52N_c n_f - 10.7n_f/N_c$ is finite [6][5] and in the $\overline{\mathrm{MS}}$ factorization scheme $\chi_1(\frac{1}{2})$ is effectively (see eq.(23)) $\chi_1^{FL}(\frac{1}{2}) + b_0 N_c[(4\ln 2)^2 - \frac{2\pi^2}{3}]$.

It follows that asymptotically at large Ξ in the $\overline{\mathrm{MS}}$ scheme

$$\frac{P_1(\Xi)}{P_0(\Xi)} \underset{\Xi\to\infty}{\sim} \frac{\chi_1(\frac{1}{2})}{\chi_0(\frac{1}{2})}\Xi + O(1): \tag{27}$$

the next-to-leading correction, despite being suppressed by a factor of α_s, rises linearly with Ξ and hence with $\ln 1/x$ at small x, and becomes eventually dominant. In terms of the anomalous dimension γ, this means that the ratio $A_1^{(n)}/A_0^{(n)}$ of the NLLx coefficients to the LLx coefficients in the expansions (3) rises linearly with n. It is clear that the origin of the rise is the simple pole of γ_1 eq. (14), viewed as a function of γ_0, at $\gamma_0 = \frac{1}{2}$. Because the denominator of eq. (14) vanishes linearly at γ_0, this simple pole is present whenever $\chi_1(\frac{1}{2})$ has a finite nonzero value.

In factorization schemes where $\chi_1(M)$ diverges at $M = 1/2$ the singularity in γ_1 will be stronger, and so P_1 will rise more rapidly at large ξ. Such factorization schemes have been considered in the literature as being possibly more appropriate at small x: in particular, the Q_0–schemes [21], the SDIS scheme [23] and the "physical" scheme (GDIS) [24,19], all of which have the advantage of reducing the size of the leading perturbative corrections in the quark sector. This reduction is accomplished at the expense of introducing a singularity in $\chi_1(M)$ at $M = \frac{1}{2}$: for instance in the Q_0–scheme (i.e. eq.(18)) $\chi_1(M)$ has a simple pole at $M = \frac{1}{2}$ due to the vanishing of $\chi_0'(\frac{1}{2})$.

It is easy to see that if the NLO anomalous dimension has a simple pole at the location of the LO saddle, this dominates the asymptotic behaviour of integral (26). In this case, on top of the contribution of eq. (26), there is a further contribution from the residue of the pole:

$$P_1^{\text{sing.}}(\Xi) \underset{\Xi\to\infty}{\sim} -\tfrac{1}{2}\mathrm{Res}[\chi_1] e^{\Xi} + \ldots, \tag{28}$$

where $\mathrm{Res}[\chi_1]$ is the residue of the simple pole of $\chi_1(M)$ at $M = \frac{1}{2}$: it is equal to $\frac{1}{2}b_0\chi_0(\frac{1}{2})$ in all of the singular schemes mentioned above. The ratio P_1/P_0 now rises as[6] $\Xi^{3/2}$:

$$\frac{P_1(\Xi)}{P_0(\Xi)} \underset{\Xi\to\infty}{\sim} \left(\frac{\chi_1^{FL}(\frac{1}{2})}{\chi_0(\frac{1}{2})} + k\right)\Xi - b_0\left(\frac{\pi\chi_0''(\frac{1}{2})}{8\chi_0(\frac{1}{2})}\right)^{1/2} \Xi^{3/2}\left[1 - \frac{\chi_0(\frac{1}{2})\chi_0''''(\frac{1}{2})}{8(\chi_0''(\frac{1}{2}))^2}\frac{1}{\Xi}\right] + O(1), \tag{29}$$

[5] Note that the results given in both [6] and [25] are numerically incorrect.

[6] A similar result was found in ref.[13].

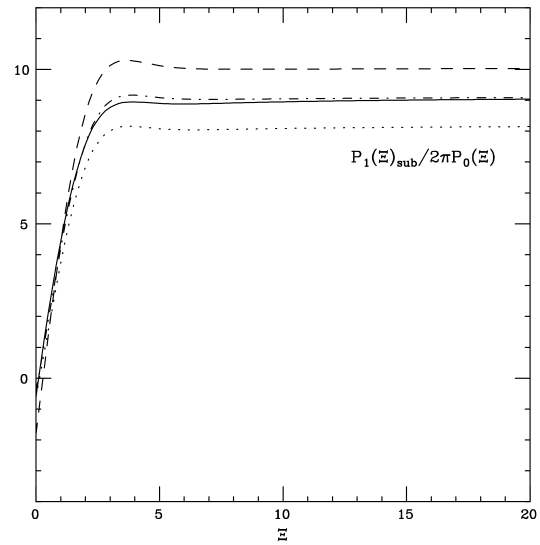

Figure 4: As fig. 3, but with the asymptotic behaviours (27) and (29) calculated analytically subtracted from the previous results computed numerically. Note the change in scale on the vertical axis.

where $k = 2b_0, -2b_0, 0$ in the Q_0-DIS [21], SDIS [23] and GDIS [24,19] schemes respectively. In all such schemes, the small x expansion appears thus to be particularly badly behaved.

In Fig. 4 the splitting function ratio P_1/P_0 is plotted again but now with the asymptotic results (27) and (29) subtracted. It is clear that the asymptotic behaviour sets in surprisingly quickly: already at $\Xi \gtrsim 3$ the subtracted ratio becomes constant. It follows that what is left of P_1 after the subtraction is no longer unnaturally large: the asymptotic growth of the ratios (27) and (29) is entirely responsible for the breakdown of small x perturbation theory at NLLx. This could have been anticipated even before the calculation of [6]: the NLLx correction to the LLx splitting function will inevitably become large asymptotically unless the NLLx correction to $\chi(\frac{1}{2})$ vanishes.

Pursuing the argument to higher orders, it is apparent that the anomalous dimensions γ_i have higher order poles as the order of the expansion increases: γ_2 eq. (15) has a triple pole, and in general γ_n has a $(2n-1)$–th order pole. Consequently, the associated splitting functions will display stronger and stronger rises with ξ. It follows that the small x expansion eqns. (2),(4) inevitably breaks down at small x: as Ξ grows, the higher orders of the expansion become more and more important. This means that the leading contributions at small x have not been properly resummed.

The origin of this failure can be simply understood by recalling that at fixed coupling the leading asymptotic small x behaviour of the solution to the ξ-evolution eq. (7) is given by $x^{a\chi(M_s)}$ where M_s is the position of the saddle point in M (so at leading order $\chi = \chi_0$ and $M_s = \frac{1}{2}$). When solving the small x Altarelli-Parisi equation eq. (1), this growth at small x, rather than being generated by solving an evolution equation in ξ eq. (7) (with ξ-independent anomalous dimensions), is included in the splitting functions. But expanding the corrections to the LLx asymptotic behaviour in powers of α_s

$$x^{a(\chi_0(M_s)+a\chi_1(M_s)+\ldots)} = x^{a\chi_0(\frac{1}{2})}\left[1 + a\Xi\frac{\chi_1(\frac{1}{2})}{\chi_0(\frac{1}{2})} + \ldots\right], \tag{30}$$

it is clear that the LLx asymptotic behaviour can be modified by subleading terms only if these rise with Ξ. For instance, the NLLx correction can only be generated if P_1/P_0 rises linearly with Ξ, with slope $\chi_1(\frac{1}{2})/\chi_0(\frac{1}{2})$, as indeed we found above in eq. (27). At higher orders in α_s the LLx behaviour receives corrections proportional to higher powers of Ξ, and correspondingly the higher order P_n have higher order poles. However these corrections are no longer given by a trivial exponentiation of the NLLx result.

The bad behaviour of the small x expansion is thus due to the fact that the asymptotic behaviour of the solution to the evolution equations at large Ξ does not coincide with the LLx prediction. The subsequent mismatch in the order of subleading corrections makes a nonsense of the perturbative expansion (3) and (5). This can only be corrected by suitably reorganizing the expansion (9) in order to properly resum the large corrections.

Since a change in the factorization scheme mixes different orders in the perturbative expansion, different scheme choices may be thought of as resummations. Furthermore just as there are scheme choices which make the perturbative expansion less stable, so there are choices which can improve it. More precisely, we can resum the large corrections by choosing the scheme in such a way that the anomalous dimensions eqns. (14),(15),... are all regular at $\gamma_0 = \frac{1}{2}$ order by order. This is always possible because if all γ_i with $i \leq i_0 - 1$ are regular at $\gamma_0 = \frac{1}{2}$, then γ_{i_0} has a simple pole at $\gamma_0 = \frac{1}{2}$ (assuming that all $\chi_i(M)$ are regular there), which can be removed by choosing a scheme which subtracts a constant from $\chi_i(M)$ equal to the numerators of eqns. (14),(15) and their higher order generalizations. Necessary conditions for a satisfactory perturbative scheme are thus that in the new scheme

$$\chi_1(\tfrac{1}{2}) = 0, \qquad \chi_2(\tfrac{1}{2}) = \tfrac{1}{2}(\chi_1'(\tfrac{1}{2}))^2/\chi_0''(\tfrac{1}{2}), \qquad \cdots. \tag{31}$$

Clearly these conditions are not very restrictive. In fact, since there is only one condition at each perturbative order, they can be imposed simply by a choice of renormalization scale, *i.e.* by the replacement of $\alpha_s(Q^2)$ with $\alpha_s((kQ)^2)$, where k may itself be expanded as a series in α_s. Then for example at NLLx $A_1^{(n)} \to A_1^{(n)} + A_0^{(n)} n b_0 \log k^2$, and the linear rise of $A_1^{(n)}/A_0^{(n)}$ in $\overline{\rm MS}$ factorization schemes may be eliminated by choosing $b_0 \log k = -\frac{1}{2}\chi_1(\frac{1}{2})/\chi_0(\frac{1}{2})$, which gives $k \simeq 300$. Choosing such a large scale does indeed lead to stable perturbative behaviour (see fig. 5). However it is also clearly a fine tuning: varying the scale by a factor of two either side leads to huge variations in the relative size of the perturbative correction. Other scale choices, such as BLM, designed to reduce the size of subleading corrections have been considered in ref.[26].[7] At NLLx it is even possible to fine tune the choice of scheme such that $\chi_1(M) \equiv 0$, by choosing the gluon normalization factor $u(M)$ as a solution to the first order differential equation $(\ln u)' = \chi_1(M)/b_0\chi_0(M)$ (cf. (20)): then $\gamma_1(\alpha_s/N) \equiv 0$ and all NLLx corrections have been removed from the large eigenvalue of evolution.

[7] It is also possible to remove the singularity (28) in singular schemes such as Q_0-DIS by tuning the scale. However now the choice of scale depends on x: $k \sim k_1 \exp(k_2\sqrt{\xi} + k_3/\sqrt{\xi})$, where each of k_1, k_2 and k_3 all require fine tuning if the perturbative expansion is to be stable. Such a scale might also be justified through a BLM prescription [27].

The physical meaning of these scheme choices is that, after the scheme change, the large corrections to the LLx asymptotic small x behaviour are absorbed into the x-dependent initial condition to the perturbative evolution in Q^2. However because these large corrections have a large scheme dependence, it seems pointless to consider them at any finite order: all contributions to the asymptotic behaviour should be resummed to all orders, and then included in the LLx anomalous dimension. To this purpose it is sufficient to subtract Q^2-independent contributions in eq. (9):

$$\chi(M;a) = c(a) + \tilde{\chi}_0(M) + a\tilde{\chi}_1(M) + \ldots, \tag{32}$$

where $\tilde{\chi}_i(M) \equiv \chi_i(M) - c_i$, $i = 0, 1, 2, \ldots$ and formally $c(a) \equiv c_0 + \sum_{n=1}^{\infty} c_n a^n$. The subtractions $c_1, c_2, \ldots$ are then fixed by the criteria (31): we need

$$c_1 = \chi_1(\tfrac{1}{2}), \qquad c_2 = \chi_2(\tfrac{1}{2}) - \tfrac{1}{2}(\chi_1'(\tfrac{1}{2}))^2/\chi_0''(\tfrac{1}{2}), \qquad \cdots. \tag{33}$$

Since the resummed splitting function will be independent of c_0, there is in principle no need to fix its value; however it is convenient to choose $c_0 = \chi_0(1/2)$ as with this choice the parameter

$$\lambda = ac(a) \tag{34}$$

has a direct physical meaning.

We can now use the expansion (32) of χ in eq.(12) to determine $\tilde{\gamma}_i$ order by order, treating $c(a)$ as leading order. At LLx the 'resummed' anomalous dimension $\tilde{\gamma}_0(N;a)$ is then the solution of

$$c(a) + \tilde{\chi}_0[\tilde{\gamma}_0] = N/a. \tag{35}$$

This implies that $\tilde{\gamma}_0(N;a) = \gamma_0[a/(N-(\lambda-ac_0))]$ and consequently that the LLX splitting function

$$\tilde{P}_0(x;a) = P_0(\Xi)e^{(\lambda-ac_0)\xi}. \tag{36}$$

In particular the asymptotic behaviour (25) becomes

$$\tilde{P}_0(x;a) \underset{\Xi\to\infty}{\sim} \left(\frac{\chi_0(\frac{1}{2})}{2\pi\chi_0''(\frac{1}{2})}\right)^{1/2} \frac{\chi_0(\frac{1}{2})}{\Xi^{3/2}}\, e^{\lambda\xi} \left[1 + O\left(\frac{1}{\Xi}\right)\right]. \tag{37}$$

The parameter λ thus determines the nature of the asymptotic small x behaviour. The expansion in powers of α_s is now well-behaved: due to the conditions (33) all higher order $P_i(x)$ behave in the same way as P_0 at large ξ, and thus all the corrections to P_0 are down by powers of α_s uniformly in x. The ratio $\tilde{P}_1/\tilde{P}_0$ is shown in fig. 5: it is indeed uniformly bounded and not too unreasonably large.[8]

[8] In fact it is identical to the subtracted ratio in DIS plotted in fig. 4: this is because

$$\frac{\chi_1(\frac{1}{2})}{\chi_0(\frac{1}{2})}\Xi P_0(\Xi) = -\chi_1(\tfrac{1}{2}) \int_{-i\infty}^{i\infty} \frac{d\gamma_0}{2\pi i}\gamma_0 \frac{\partial}{\partial\gamma_0} e^{a\xi\chi_0(\gamma_0)} = c_1 \int_{-i\infty}^{i\infty} \frac{dN}{2\pi i a} \frac{e^{N\xi}}{(-\chi_0'(\gamma_0(a/N)))},$$

by an integration by parts and change of variables.

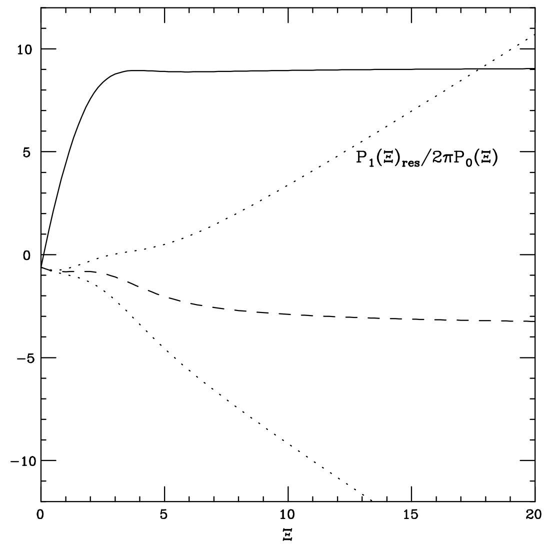

Figure 5: As fig. 3, but after various resummations: the subtraction (32), the fine tuned scale (dashed), and scales a factor of two either side of it (dotted).

The reorganization of the perturbative expansion (32) can be viewed as an effective resummation of the higher orders of the expansion. Since $c(a) - c_0$ is of $O(a)$, eq. (36) implies that $\tilde{P}_0$ and P_0 differ by a series of formally subleading contributions. However, the parameter λ (34) which summarizes the asymptotic behaviour at small x is treated as α_s-independent, and thus included in the leading order eq.(35). This 'order transmutation' effectively resums into the LLx anomalous dimension the all-order behaviour as given by λ. However, to determine the value of λ it may be necessary to use arguments which go beyond mass factorised perturbation theory. In particular, the unitarity constraint would suggest that $\lambda \leq 0$: if λ were positive the resulting powerlike growth in the splitting function at small x would drive a corresponding rise in the cross-section, which would ultimately violate the Froissart bound.

The removal of the unbounded growth of formally subleading corrections at small x, achieved by the resummation described above, is a necessary prerequisite for a consistent small x resummation of Altarelli-Parisi evolution. Although our resummation does not resolve the instability in the small x evolution equation discussed in [11-14], it does show that this is a separate issue. Indeed, the instability is clearly related to the shape of χ as $M \to 0$ and $M \to 1$, whereas the resummation criteria (31) refer to $M = \frac{1}{2}$. At small M (and thus large Q^2), the relevant approximation is to use the conventional Altarelli-Parisi equation: from this it may readily be inferred (using a duality argument [16] inverse to that used to obtain (12)) that the resummed kernel must always be finite and positive at $M = 0$, which is probably sufficient to cure the instability. Possibly related attempts to deal with these instabilities have been presented in ref.[28].

To conclude, we have shown that the poor behaviour of the small x expansion which is manifested [9,10] in the NLLx splitting functions computed from the recent Fadin-Lipatov determination [6] of the next-to-leading high energy QCD asymptotics can be traced to the fact that the formally NLLx corrections to the LLx contributions to the splitting

functions are not truly subleading at small x. We have shown that this problem persists to all orders, and is related to the fact that the leading small x behaviour is not given by the leading order term of the small x expansion, but rather must come from an all-order resummation. We have demonstrated that a reorganization of the perturbative expansion is necessary, and given criteria eq.(31) which must be met if such a resummation is to be successful. We further constructed a resummation which meets these criteria, but depends on a new parameter λ eq.(34) which controls the asymptotic growth at small x. A complete resummation of Altarelli-Parisi splitting functions at small x might now be achieved through careful matching in the high Q^2 region.

Acknowledgements: We thank G. Altarelli for several discussions and a critical reading of the manuscript. This work was supported in part by a PPARC Visiting Fellowship, and EU TMR contract FMRX-CT98-0194 (DG 12 - MIHT).

References

[1] See *e.g.* R.D. Ball and A. DeRoeck, `hep-ph/9609309`;
S. Forte, `hep-ph/9812382` and ref. therein.

[2] A. De Rújula et al., *Phys. Rev.* **10** (1974) 1649.

[3] R.D. Ball and S. Forte, *Phys. Lett.* **B335** (1994) 77.

[4] R.D. Ball and S. Forte, *Phys. Lett.* **B351** (1995) 313;
R.K. Ellis, F. Hautmann and B. R. Webber, *Phys. Lett.* **B348** (1995) 582.

[5] R.D. Ball and S. Forte, `hep-ph/9607291`;
see also I. Bojak and M. Ernst, *Nucl. Phys.* **B508** (1997) 731.

[6] V. S. Fadin and L. N. Lipatov, *Phys. Lett.* **B429** (1998) 127.

[7] V. S. Fadin et al, *Phys. Lett.* **B359** (1995) 181; *Phys. Lett.* **B387** (1996) 593; *Nucl. Phys.* **B406** (1993) 259; *Phys. Rev.* **D50** (1994) 5893; *Phys. Lett.* **B389** (1996) 737; *Nucl. Phys.* **B477** (1996) 767; *Phys. Lett.* **B415** (1997) 97; *Phys. Lett.* **B422** (1998) 287.

[8] V. del Duca, *Phys. Rev.* **D54** (1996) 989; *Phys. Rev.* **D54** (1996) 4474 ;
V. del Duca and C. R. Schmidt, *Phys. Rev.* **D57** (1998) 4069 ;
Z. Bern, V. del Duca and C. R. Schmidt, *Phys. Lett.* **B445** (1998) 168 ;
G. Camici and M. Ciafaloni, *Phys. Lett.* **B412** (1997) 396; *Phys. Lett.* **B430** (1998) 349.

[9] R. D. Ball and S. Forte, `hep-ph/9805315`.

[10] J. Blümlein et al., `hep-ph/9806368`.

[11] D. A. Ross, *Phys. Lett.* **B431** (1998) 161.

[12] E. Levin, `hep-ph/9806228`.

[13] Y. V. Kovchegov and A. H. Mueller, *Phys. Lett.* **B439** (1998) 428.

[14] N. Armesto, J. Bartels and M.A. Braun, *Phys. Lett.* **B442** (1998) 459.

[15] T. Jaroszewicz, *Phys. Lett.* **B116** (1982) 291;
S. Catani, F. Fiorani and G. Marchesini, *Phys. Lett.* **B336** (1990) 18 ;
S. Catani et al., *Nucl. Phys.* **B361** (1991) 645.

[16] R. D. Ball and S. Forte, *Phys. Lett.* **B405**, 317 (1997).

[17] For a review, see V. del Duca, `hep-ph/9503226` and ref. therein.

[18] G. Camici and M. Ciafaloni, *Nucl. Phys.* **B496** (1997) 305.

[19] R. D. Ball and S. Forte, *Phys. Lett.* **B359**, 362 (1995).

[20] S. Catani and F. Hautmann, *Phys. Lett.* **B315** (1993) 157; *Nucl. Phys.* **B427** (1994) 475.

[21] M. Ciafaloni, *Phys. Lett.* **B356** (1995) 74.

[22] R.D. Ball and S. Forte, unpublished.

[23] S. Catani, *Zeit. Phys.* **C70** (1996) 263.

[24] S. Catani, *Z. Phys.* **C75**, 665 (1997).

[25] V.S. Fadin, hep-ph/9807527

[26] S.J. Brodsky et al, hep-ph/9901229.

[27] R.S. Thorne, hep-ph/9901331.

[28] G. Salam, *Jour. High Energy Phys.* **9807** (1998) 19;
M. Ciafaloni and D. Colferai, hep-ph/9812366;
C.R. Schmidt, hep-ph/9901397;
J.R. Forshaw, D.A. Ross and A. Sabio Vera, hep-ph/9903390.